

FACTA UNIVERSITATIS (NIŠ)
 SER. MATH. INFORM. Vol. 32, No 2 (2017), 195–207
 DOI: 10.22190/FUMI1702195F

APPROXIMATION THEOREMS FOR LIMIT (p, q) –BERNSTEIN-DURRMEYER OPERATOR

Zoltan Finta and Vijay Gupta

Abstract. In the present paper, using the method developed in [6], we prove the existence of the limit operator of the slight modification of the sequence of (p, q) -Bernstein-Durrmeyer operators introduced recently in [10]. We also establish the rate of convergence of this limit operator.

Keywords: Approximation theorems; (p, q) –Bernstein-Durrmeyer operator; Rate of convergence

1. Introduction

The applications of q -calculus in the field of approximation theory have led to the discovery of new generalizations of Bernstein operators. The first generalization involving q -integers was obtained by Lupaş [13] in 1987. Ten years later Phillips [18] gave another generalization of the Bernstein operators introducing the so-called q -Bernstein operators. After that, several well-known positive linear operators and other new operators have been generalized to their q -variants and their approximation behavior have been studied (see e.g. [3] and [11]). The concept of the limit q -Bernstein operator was introduced by Il'inskiĭ and Ostrovska [12], and its rate of convergence was established by Wang and Meng [23], and Finta [7], respectively. Nowadays, the (p, q) -calculus renders to find new generalizations of q -Bernstein operators possible (see [1], [2], [4], [16], [22], [15], [8], [14]). Some basic definitions and theorems of (p, q) -calculus may be found in the papers [9], [21], [19] and [20].

The (p, q) -integers $[n]_{p,q}$ are defined by

$$[n]_{p,q} = \frac{p^n - q^n}{p - q}, \quad q \neq p.$$

For $p = 1$, we recover the well-known q -integers $[n]_q = (1 - q^n)/(1 - q)$. Obviously

$$(1.1) \quad [n]_{p,q} = p^{n-1} [n]_{q/p}.$$

Received September 16, 2016; accepted January 05, 2017

2010 *Mathematics Subject Classification.* Primary 41A25; Secondary 41A30, 41A36

The (p, q) -factorials $[n]_{p,q}!$ are defined by

$$[n]_{p,q}! = \begin{cases} [1]_{p,q}[2]_{p,q} \cdots [n]_{p,q}, & \text{if } n \geq 1 \\ 1, & \text{if } n = 0, \end{cases}$$

and the (p, q) -binomial coefficients are given by

$$\begin{bmatrix} n \\ k \end{bmatrix}_{p,q} = \frac{[n]_{p,q}!}{[k]_{p,q}![n-k]_{p,q}!}, \quad 0 \leq k \leq n.$$

Further, the (p, q) -power basis is defined by

$$(x \ominus a)_{p,q}^n = \begin{cases} (x-a)(px-qa)(p^2x-q^2a) \cdots (p^{n-1}x-q^{n-1}a), & \text{if } n \geq 1 \\ 1, & \text{if } n = 0, \end{cases}$$

and the (p, q) -integral of f over the interval $[0, a]$ is defined as

$$\int_0^a f(t) d_{p,q}t = (p-q)a \sum_{k=0}^{\infty} \frac{q^k}{p^{k+1}} f\left(\frac{aq^k}{p^{k+1}}\right) \quad \text{for } 0 < q < p \leq 1.$$

In [10] the (p, q) -analogue of the Bernstein-Durrmeyer operators was introduced in the following way:

$$(1.2) \quad D_n^{p,q}(f; x) = [n+1]_{p,q} \sum_{k=0}^n p^{-(n^2+3n-k^2-k)/2} b_{n,k}^{p,q}(1, x) \int_0^1 \tilde{b}_{n,k}^{p,q}(p, pqt) f(t) d_{p,q}t,$$

where $f \in C[0, 1]$, $x \in [0, 1]$, $0 < q < p \leq 1$,

$$b_{n,k}^{p,q}(1, x) = \begin{bmatrix} n \\ k \end{bmatrix}_{p,q} p^{(k(k-1)-n(n-1))/2} x^k (1 \ominus x)_{p,q}^{n-k}$$

and

$$\tilde{b}_{n,k}^{p,q}(p, pqt) = \begin{bmatrix} n \\ k \end{bmatrix}_{p,q} (pt)^k (p \ominus pqt)_{p,q}^{n-k}.$$

In what follows we propose the following slight modification of (1.2):

$$(1.3) \quad \tilde{D}_n^{p,q}(f; x) = [n+1]_{p,q} \sum_{k=0}^n p^{-(n^2+3n-k^2-k)/2} b_{n,k}^{p,q}(1, x) \int_0^1 \tilde{b}_{n,k}^{p,q}(p, pqt) f(pt) d_{p,q}t,$$

where $b_{n,k}^{p,q}(1, x)$ and $\tilde{b}_{n,k}^{p,q}(p, pqt)$ have got the same expressions as above. For $p = q = 1$, we recover the Durrmeyer operators (see [5]). The goal of the paper is to study the limit (p, q) -Bernstein-Durrmeyer operator $\tilde{D}_{\infty}^{p,q} : C[0, 1] \rightarrow C[0, 1]$ defined

by $\tilde{D}_\infty^{p,q}(f; x) = \lim_{n \rightarrow \infty} \tilde{D}_n^{p,q}(f; x)$, where $f \in C[0, 1]$ is arbitrary. Throughout the paper we fix the parameters p, q such that $0 < q < p \leq 1$. We establish the rate of convergence of $\tilde{D}_\infty^{p,q}(f; x)$ using the modulus of continuity of $f \in C[0, 1]$ given by

$$(1.4) \quad \omega(f, \delta) = \sup\{|f(x) - f(y)| : x, y \in [0, 1], |x - y| \leq \delta\}, \quad \delta > 0.$$

The existence of $\tilde{D}_\infty^{p,q}$ is proven with the aid of the method developed in [6]. More precisely, we shall apply the following result (see [6, p. 393, Theorem 2.1] and [6, p. 394, Corollary 2.1]):

Theorem 1.1. *Let Λ be a set of parameters and for $\lambda \in \Lambda$ let $(L_n^\lambda)_{n \geq 1}$ be a sequence of positive linear operators on $C[0, 1]$. If there exist the positive sequences $(\alpha_n)_{n \geq 1}$ and $(\beta_n)_{n \geq 1}$ such that*

- a) $\alpha_n \rightarrow 0$ as $n \rightarrow \infty$,
- b) there exists $C_1 > 0$ with $\beta_n + \beta_{n+1} + \dots + \beta_{n+m-1} \leq C_1 \alpha_n$ for all $n, m \geq 1$,
- c) there exists $C_2 > 0$ with $\|L_n^\lambda g - L_{n+1}^\lambda g\| \leq C_2 \beta_n \|g'\|$ for all $n \geq 1$ and $g \in C^1[0, 1]$,

then there exists $C_3 = C_3(\|L_1^\lambda e_0\|) > 0$ and a positive linear operator $L_\infty^\lambda : C[0, 1] \rightarrow C[0, 1]$ such that

$$\|L_n^\lambda f - L_\infty^\lambda f\| \leq C_3 \omega(f, \alpha_n)$$

for all $f \in C[0, 1]$ and $n = 1, 2, \dots$

We mention that $\|\cdot\|$ denotes the uniform norm on $C[0, 1]$, $e_0(x) = 1$ for $x \in [0, 1]$, and the sequences $(\alpha_n)_{n \geq 1}$ and $(\beta_n)_{n \geq 1}$ may depend on λ .

2. Main results

First, we establish some auxiliary results.

Lemma 2.1. *With the notation*

$$\lambda_{n,k}^{p,q}(f) = [n+1]_{p,q} p^{-(n^2+3n-k^2-k)/2} \int_0^1 \tilde{b}_{n,k}^{p,q}(p, pqt) f(pt) d_{p,q}t,$$

where $k = 0, 1, \dots, n$ and $f \in C[0, 1]$, we have for $x \in [0, 1]$ that

$$\begin{aligned} & \tilde{D}_n^{p,q}(f; x) - \tilde{D}_{n+1}^{p,q}(f; x) \\ &= b_{n+1,0}^{p,q}(1, x) \{ \lambda_{n,0}^{p,q}(f) - \lambda_{n+1,0}^{p,q}(f) \} + \sum_{k=1}^n b_{n+1,k}^{p,q}(1, x) \left\{ \lambda_{n,k}^{p,q}(f) \frac{[n+1-k]_{p,q}}{[n+1]_{p,q}} p^k \right. \\ & \quad \left. + \lambda_{n,k-1}^{p,q}(f) \frac{[k]_{p,q}}{[n+1]_{p,q}} q^{n+1-k} - \lambda_{n+1,k}^{p,q}(f) \right\} + b_{n+1,n+1}^{p,q}(1, x) \{ \lambda_{n,n}^{p,q}(f) - \lambda_{n+1,n+1}^{p,q}(f) \}. \end{aligned}$$

Proof. We express the difference $\tilde{D}_n^{p,q}(f; x) - \tilde{D}_{n+1}^{p,q}(f; x)$ as follows (see also [17, pp. 411-412]):

$$\begin{aligned}
 & \prod_{l=0}^n (p^l - q^l x)^{-1} \{ \tilde{D}_n^{p,q}(f; x) - \tilde{D}_{n+1}^{p,q}(f; x) \} \\
 &= \sum_{k=0}^n \lambda_{n,k}^{p,q}(f) \left[\begin{matrix} n \\ k \end{matrix} \right]_{p,q} p^{(k(k-1)-n(n-1))/2} x^k \prod_{l=n-k}^n (p^l - q^l x)^{-1} \\
 (2.1) \quad & - \sum_{k=0}^{n+1} \lambda_{n+1,k}^{p,q}(f) \left[\begin{matrix} n+1 \\ k \end{matrix} \right]_{p,q} p^{(k(k-1)-(n+1)n)/2} x^k \prod_{l=n+1-k}^n (p^l - q^l x)^{-1}.
 \end{aligned}$$

Because

$$\begin{aligned}
 x^k \prod_{l=n-k}^n (p^l - q^l x)^{-1} &= x^k p^{-n+k} (p^{n-k} - q^{n-k} x + q^{n-k} x) \prod_{l=n-k}^n (p^l - q^l x)^{-1} \\
 &= p^{-n+k} x^k \prod_{l=n+1-k}^n (p^l - q^l x)^{-1} + \left(\frac{q}{p} \right)^{n-k} x^{k+1} \prod_{l=n-k}^n (p^l - q^l x)^{-1},
 \end{aligned}$$

we get from (2.1), that

$$\begin{aligned}
 & \prod_{l=0}^n (p^l - q^l x)^{-1} \{ \tilde{D}_n^{p,q}(f; x) - \tilde{D}_{n+1}^{p,q}(f; x) \} \\
 &= \sum_{k=0}^n \lambda_{n,k}^{p,q}(f) \left[\begin{matrix} n \\ k \end{matrix} \right]_{p,q} p^{(k(k-1)-n(n-1))/2} p^{-n+k} x^k \prod_{l=n+1-k}^n (p^l - q^l x)^{-1} \\
 &+ \sum_{k=0}^n \lambda_{n,k}^{p,q}(f) \left[\begin{matrix} n \\ k \end{matrix} \right]_{p,q} p^{(k(k-1)-n(n-1))/2} \left(\frac{q}{p} \right)^{n-k} x^{k+1} \prod_{l=n-k}^n (p^l - q^l x)^{-1} \\
 &- \sum_{k=0}^{n+1} \lambda_{n+1,k}^{p,q}(f) \left[\begin{matrix} n+1 \\ k \end{matrix} \right]_{p,q} p^{(k(k-1)-(n+1)n)/2} x^k \prod_{l=n+1-k}^n (p^l - q^l x)^{-1} \\
 &= \{ \lambda_{n,0}^{p,q}(f) p^{-n(n-1)/2} p^{-n} - \lambda_{n+1,0}^{p,q}(f) p^{-(n+1)n/2} \} \\
 &+ \sum_{k=1}^n \left[\begin{matrix} n+1 \\ k \end{matrix} \right]_{p,q} p^{(k(k-1)-(n+1)n)/2} x^k \prod_{l=n+1-k}^n (p^l - q^l x)^{-1} \\
 &\times \left\{ \lambda_{n,k}^{p,q}(f) \frac{\left[\begin{matrix} n \\ k \end{matrix} \right]_{p,q}}{\left[\begin{matrix} n+1 \\ k \end{matrix} \right]_{p,q}} p^k + \lambda_{n,k-1}^{p,q}(f) \frac{\left[\begin{matrix} n \\ k-1 \end{matrix} \right]_{p,q}}{\left[\begin{matrix} n+1 \\ k \end{matrix} \right]_{p,q}} p^{n+1-k} \left(\frac{q}{p} \right)^{n+1-k} - \lambda_{n+1,k}^{p,q}(f) \right\} \\
 &+ \{ \lambda_{n,n}^{p,q}(f) - \lambda_{n+1,n+1}^{p,q}(f) \} x^{n+1} \prod_{l=0}^n (p^l - q^l x)^{-1}
 \end{aligned}$$

$$\begin{aligned}
 &= \{ \lambda_{n,0}^{p,q}(f) - \lambda_{n+1,0}^{p,q}(f) \} p^{-(n+1)n/2} \\
 &\quad + \sum_{k=1}^n \left[\begin{matrix} n+1 \\ k \end{matrix} \right]_{p,q} p^{(k(k-1)-(n+1)n)/2} x^k \prod_{l=n+1-k}^n (p^l - q^l x)^{-1} \\
 &\quad \times \left\{ \lambda_{n,k}^{p,q}(f) \frac{[n+1-k]_{p,q}}{[n+1]_{p,q}} p^k + \lambda_{n,k-1}^{p,q}(f) \frac{[k]_{p,q}}{[n+1]_{p,q}} q^{n+1-k} - \lambda_{n+1,k}^{p,q}(f) \right\} \\
 &\quad + \{ \lambda_{n,n}^{p,q}(f) - \lambda_{n+1,n+1}^{p,q}(f) \} x^{n+1} \prod_{l=0}^n (p^l - q^l x)^{-1}.
 \end{aligned}$$

Multiplying with $\prod_{l=0}^n (p^l - q^l x)$, we get the assertion of the lemma. \square

Lemma 2.2. For

$$\tilde{b}_{n,k}^{p,q}(p, pqt) = \left[\begin{matrix} n \\ k \end{matrix} \right]_{p,q} (pt)^k (p \ominus pqt)_{p,q}^{n-k}, \quad k = 0, 1, \dots, n,$$

we have

$$\begin{aligned}
 [n+1]_{p,q} p^{-(n^2+3n-k^2-k)/2} \int_0^1 \tilde{b}_{n,k}^{p,q}(p, pqt) d_{p,q}t &= 1, \\
 [n+1]_{p,q} p^{-(n^2+3n-k^2-k)/2} \int_0^1 \tilde{b}_{n,k}^{p,q}(p, pqt) t d_{p,q}t &= p^{n-k} \frac{[k+1]_{p,q}}{[n+2]_{p,q}}, \\
 [n+1]_{p,q} p^{-(n^2+3n-k^2-k)/2} \int_0^1 \tilde{b}_{n,k}^{p,q}(p, pqt) t^2 d_{p,q}t &= p^{2(n-k)} \frac{[k+1]_{p,q} [k+2]_{p,q}}{[n+2]_{p,q} [n+3]_{p,q}}.
 \end{aligned}$$

Proof. The equalities follow from the computations used in [10, Lemma 3.1]. \square

Lemma 2.3. For $x \in [0, 1]$, we have

$$\begin{aligned}
 \tilde{D}_n^{p,q}(1; x) &= 1, \quad \tilde{D}_n^{p,q}(t; x) = \frac{p^{n+1} + pq[n]_{p,q}x}{[n+2]_{p,q}}, \\
 \tilde{D}_n^{p,q}(t^2; x) &= \frac{p^{2n+2}[2]_{p,q}}{[n+2]_{p,q}[n+3]_{p,q}} + \frac{(2q^2 + qp)p^{n+2}[n]_{p,q}x}{[n+2]_{p,q}[n+3]_{p,q}} \\
 &\quad + \frac{q^3[n]_{p,q}[p^2[n]_{p,q}x^2 + p^{n+1}x(1-x)]}{[n+2]_{p,q}[n+3]_{p,q}}.
 \end{aligned}$$

Proof. Analogously to the proof of [10, Lemma 3.1], we get the statements of the lemma. \square

Remark 2.1. If $p = p(n)$ and $q = q(n)$ such that $0 < q(n) < p(n) \leq 1$ and $q(n) \rightarrow 1$ as $n \rightarrow \infty$, then, by Korovkin's theorem, $\tilde{D}_n^{p,q}(f; x)$ converges uniformly to $f(x)$ for $x \in [0, 1]$,

as $n \rightarrow \infty$. Indeed, for each n the estimates

$$\begin{aligned} |\tilde{D}_n^{p,q}(t; x) - x| &\leq \frac{1}{[n+2]_{q/p}} + \left| \frac{q}{p} \frac{[n]_{q/p}}{[n+2]_{q/p}} - 1 \right|, \\ |\tilde{D}_n^{p,q}(t^2; x) - x^2| &\leq \frac{[2]_{q/p}}{[n+2]_{q/p}[n+3]_{q/p}} + \left(2 \left(\frac{q}{p} \right)^3 + \frac{q}{p} \right) \frac{[n]_{q/p}}{[n+2]_{q/p}} \frac{1}{[n+3]_{q/p}} \\ &\quad + \left(\frac{q}{p} \right)^3 \frac{[n]_{q/p}}{[n+2]_{q/p}} \frac{1}{4[n+3]_{q/p}} + \left| \left(\frac{q}{p} \right)^3 \frac{[n]_{q/p}}{[n+2]_{q/p}} \frac{[n]_{q/p}}{[n+3]_{q/p}} - 1 \right|, \end{aligned}$$

and the facts that $[n]_{q_n/p_n} \rightarrow \infty$ and $\frac{[n]_{q_n/p_n}}{[n+2]_{q_n/p_n}} \rightarrow 1$ as $n \rightarrow \infty$, imply our statement.

In the next theorem we prove the existence of the limit (p, q) -Bernstein-Durrmeyer operator.

Theorem 2.1. *Let $\tilde{D}_n^{p,q}(f; x)$ be defined by (1.3), where p and q are fixed with $0 < q < p \leq 1$. Then there exist an absolute constant $C > 0$ and a positive linear operator $\tilde{D}_\infty^{p,q} : C[0, 1] \rightarrow C[0, 1]$ such that*

$$\|\tilde{D}_n^{p,q}(f; x) - \tilde{D}_\infty^{p,q}(f; x)\| \leq C\omega \left(f, \left(\frac{q}{p} \right)^{n/2} \right)$$

for all $f \in C[0, 1]$ and $n = 1, 2, \dots$

Proof. We have $[n+1]_{p,q} = p^k[n+1-k]_{p,q} + q^{n+1-k}[k]_{p,q}$ for $k = 0, 1, \dots, n+1$. Using the notation of Lemma 2.1, we obtain for $f \in C[0, 1]$, that

$$\begin{aligned} &\lambda_{n,k}^{p,q}(f) \frac{[n+1-k]_{p,q}}{[n+1]_{p,q}} p^k + \lambda_{n,k-1}^{p,q}(f) \frac{[k]_{p,q}}{[n+1]_{p,q}} q^{n+1-k} - \lambda_{n+1,k}^{p,q}(f) \\ &= \frac{[n+1-k]_{p,q}}{[n+1]_{p,q}} p^k \{ \lambda_{n,k}^{p,q}(f) - \lambda_{n+1,k}^{p,q}(f) \} \\ (2.2) \quad &+ \frac{[k]_{p,q}}{[n+1]_{p,q}} q^{n+1-k} \{ \lambda_{n,k-1}^{p,q}(f) - \lambda_{n+1,k}^{p,q}(f) \}. \end{aligned}$$

Let $g \in C^1[0, 1]$ and $x_k = p^{n+2-k} \frac{[k+1]_{p,q}}{[n+3]_{p,q}}$ for $k = 0, 1, \dots, n$. Obviously, by (1.1), we have

$$x_k = p^{n+2-k} \frac{p^k[k+1]_{q/p}}{p^{n+2}[n+3]_{q/p}} = \frac{[k+1]_{q/p}}{[n+3]_{q/p}} \in [0, 1],$$

where $k = 0, 1, \dots, n$. Further $g(pt) = g(x_k) + \int_{x_k}^{pt} g'(u) du$, where $t \in [0, 1]$ is arbitrary. Hence, by definition of $\lambda_{n,k}^{p,q}(g)$ and Lemma 2.2, we obtain

$$\lambda_{n,k}^{p,q}(g) - \lambda_{n+1,k}^{p,q}(g)$$

$$\begin{aligned}
 &= [n+1]_{p,q} p^{-(n^2+3n-k^2-k)/2} \int_0^1 \tilde{b}_{n,k}^{p,q}(p, pqt) \left[g(x_k) + \int_{x_k}^{pt} g'(u) du \right] d_{p,q} t \\
 &\quad - [n+2]_{p,q} p^{-((n+1)^2+3(n+1)-k^2-k)/2} \\
 &\quad \int_0^1 \tilde{b}_{n+1,k}^{p,q}(p, pqt) \left[g(x_k) + \int_{x_k}^{pt} g'(u) du \right] d_{p,q} t \\
 &= [n+2]_{p,q} p^{-((n+1)^2+3(n+1)-k^2-k)/2} \int_0^1 \tilde{b}_{n+1,k}^{p,q}(p, pqt) \left(\int_{x_k}^{pt} g'(u) du \right) \\
 (2.3) \quad &\left\{ \frac{[n+1]_{p,q}}{[n+2]_{p,q}} p^{n+2} \frac{\tilde{b}_{n,k}^{p,q}(p, pqt)}{\tilde{b}_{n+1,k}^{p,q}(p, pqt)} - 1 \right\} d_{p,q} t
 \end{aligned}$$

for $k = 0, 1, \dots, n$. On the other hand, using $[n+2]_{p,q} = p^{k+1}[n+1-k]_{p,q} + q^{n+1-k}[k+1]_{p,q}$, we have

$$\begin{aligned}
 &\frac{[n+1]_{p,q}}{[n+2]_{p,q}} p^{n+2} \frac{b_{n,k}^{p,q}(p, pqt)}{b_{n+1,k}^{p,q}(p, pqt)} - 1 \\
 &= \frac{[n+1]_{p,q}}{[n+2]_{p,q}} p^{n+2} \frac{\begin{bmatrix} n \\ k \end{bmatrix}_{p,q} (pt)^k (p \ominus pqt)_{p,q}^{n-k}}{\begin{bmatrix} n+1 \\ k \end{bmatrix}_{p,q} (pt)^k (p \ominus pqt)_{p,q}^{n+1-k}} - 1 \\
 &= \frac{[n+1]_{p,q}}{[n+2]_{p,q}} p^{n+2} \frac{[n+1-k]_{p,q}}{[n+1]_{p,q}} \frac{1}{p^{n+1-k} - pq^{n+1-k}t} - 1 \\
 &= \frac{[n+1-k]_{p,q}}{[n+2]_{p,q}} \frac{p^{n+2}}{p^{n+1-k} - pq^{n+1-k}t} - 1 \\
 (2.4) \quad &= q^{n+1-k} \left(\frac{[n+1-k]_{p,q}}{[n+2]_{p,q}} \frac{p^{k+2}t}{p^{n+1-k} - pq^{n+1-k}t} - \frac{[k+1]_{p,q}}{[n+2]_{p,q}} \right).
 \end{aligned}$$

The equality $[n+2]_{p,q} = p^{k+1}[n+1-k]_{p,q} + q^{n+1-k}[k+1]_{p,q}$ implies, that

$$(2.5) \quad \frac{[n+1-k]_{p,q}}{[n+2]_{p,q}} \leq p^{-(k+1)}.$$

Analogously, the equality $[n+2]_{p,q} = q^{k+1}[n+1-k]_{p,q} + p^{n+1-k}[k+1]_{p,q}$ implies, that

$$(2.6) \quad \frac{[k+1]_{p,q}}{[n+2]_{p,q}} \leq p^{-(n+1-k)}.$$

Finally, the function $t \rightarrow p^{k+2}t/(p^{n+1-k} - pq^{n+1-k}t)$ is increasing on $[0, 1]$, therefore

$$\begin{aligned}
 \frac{p^{k+2}t}{p^{n+1-k} - pq^{n+1-k}t} &\leq \frac{p^{k+2}}{p^{n+1-k} - pq^{n+1-k}} = \frac{p^{k+2}}{p^{n+1-k} \left(1 - p\left(\frac{q}{p}\right)^{n+1-k} \right)} \\
 (2.7) \quad &\leq \frac{p^{k+2}}{p^{n+1-k} \left(1 - p\frac{q}{p} \right)} = \frac{p^{k+2}}{p^{n+1-k}(1-q)}.
 \end{aligned}$$

Combining (2.3)-(2.7), applying Lemma 2.2 and Hölder's inequality, we find

$$\begin{aligned}
& |\lambda_{n,k}^{p,q}(g) - \lambda_{n+1,k}^{p,q}(g)| \\
& \leq \|g'\| [n+2]_{p,q} p^{-[(n+1)^2+3(n+1)-k^2-k]/2} \\
& \quad \times \int_0^1 \tilde{b}_{n+1,k}^{p,q}(p, pqt) |pt - x_k| q^{n+1-k} \left(p^{-(k+1)} \frac{p^{k+2}}{p^{n+1-k}(1-q)} + p^{-(n+1-k)} \right) d_{p,q}t \\
& \leq \frac{1+p-q}{1-q} \|g'\| \left(\frac{q}{p} \right)^{n+1-k} \left\{ [n+2]_{p,q} p^{-((n+1)^2+3(n+1)-k^2-k)/2} \right. \\
& \quad \left. (2.8) \int_0^1 \tilde{b}_{n+1,k}^{p,q}(p, pqt) (pt - x_k)^2 d_{p,q}t \right\}^{1/2}.
\end{aligned}$$

Using Lemma 2.2, we get

$$\begin{aligned}
& [n+2]_{p,q} p^{-((n+1)^2+3(n+1)-k^2-k)/2} \int_0^1 \tilde{b}_{n+1,k}^{p,q}(p, pqt) (pt - x_k)^2 d_{p,q}t \\
& = p^{2(n+2-k)} \frac{[k+1]_{p,q} [k+2]_{p,q}}{[n+3]_{p,q} [n+4]_{p,q}} - 2p^{n+2-k} \frac{[k+1]_{p,q}}{[n+3]_{p,q}} p^{n+2-k} \frac{[k+1]_{p,q}}{[n+3]_{p,q}} \\
& \quad + \left(p^{n+2-k} \frac{[k+1]_{p,q}}{[n+3]_{p,q}} \right)^2 \\
& = p^{2(n+2-k)} \frac{[k+1]_{p,q}}{[n+3]_{p,q}} \left(\frac{[k+2]_{p,q}}{[n+4]_{p,q}} - \frac{[k+1]_{p,q}}{[n+3]_{p,q}} \right) \\
(2.9) \quad & = p^{2(n+2-k)} \frac{[k+1]_{p,q}}{[n+3]_{p,q}} (pq)^{k+1} \frac{[n+2-k]_{p,q}}{[n+3]_{p,q} [n+4]_{p,q}}.
\end{aligned}$$

Analogously to (2.6) and (2.5), we find that

$$(2.10) \quad \frac{[k+1]_{p,q}}{[n+3]_{p,q}} \leq p^{-(n+2-k)} \quad \text{and} \quad \frac{[n+2-k]_{p,q}}{[n+3]_{p,q}} \leq p^{-(k+1)}.$$

Further, using (1.1), we get $[n+4]_{p,q} = p^{n+3} [n+4]_{q/p} \geq p^{n+3}$. Hence, by (2.8)-(2.10), we have for $k = 0, 1, \dots, n$, that

$$\begin{aligned}
|\lambda_{n,k}^{p,q}(g) - \lambda_{n+1,k}^{p,q}(g)| & \leq \frac{1+p-q}{1-q} \|g'\| \\
& \quad \left(\frac{q}{p} \right)^{n+1-k} \left\{ p^{2(n+2-k)} p^{-(n+2-k)} (pq)^{k+1} p^{-(k+1)} p^{-(n+3)} \right\}^{1/2} \\
& = \frac{1+p-q}{1-q} \|g'\| \left(\frac{q}{p} \right)^{n+1-k} \left(\frac{q}{p} \right)^{(k+1)/2} \\
& = \frac{1+p-q}{1-q} \|g'\| \left(\frac{q}{p} \right)^{(2n-k+3)/2} \\
(2.11) \quad & \leq \frac{1+p-q}{1-q} \|g'\| \left(\frac{q}{p} \right)^{n/2} \left(\frac{q}{p} \right)^{3/2}.
\end{aligned}$$

Analogously to (2.3), we obtain for $g \in C^1[0, 1]$ and $y_k = p^{n+2-k} \frac{[k]_{p,q}}{[n+2]_{p,q}} \in [0, 1]$, where $k = 1, 2, \dots, n+1$, that

$$\begin{aligned}
 & \lambda_{n,k-1}^{p,q}(g) - \lambda_{n+1,k}^{p,q}(g) \\
 &= [n+1]_{p,q} p^{-(n^2+3n-(k-1)^2-(k-1))/2} \int_0^1 \tilde{b}_{n,k-1}^{p,q}(p, pqt) \left(\int_{y_k}^{pt} g'(u) du \right) \\
 (2.12) \quad & \times \left\{ 1 - \frac{[n+2]_{p,q}}{[n+1]_{p,q}} p^{-(n+2-k)} \frac{\tilde{b}_{n+1,k}^{p,q}(p, pqt)}{\tilde{b}_{n,k-1}^{p,q}(p, pqt)} \right\} d_{p,q}t.
 \end{aligned}$$

But

$$\tilde{b}_{n+1,k}^{p,q}(p, pqt) = \frac{[n+1]_{p,q}}{[k]_{p,q}} pt \tilde{b}_{n,k-1}^{p,q}(p, pqt)$$

for $k = 1, 2, \dots, n+1$, hence, in view of (2.12), Lemma 2.2 and Hölder's inequality, we find that

$$\begin{aligned}
 & |\lambda_{n,k-1}^{p,q}(g) - \lambda_{n+1,k}^{p,q}(g)| \\
 & \leq \|g'\| [n+1]_{p,q} p^{-(n^2+3n-(k-1)^2-(k-1))/2} \int_0^1 \tilde{b}_{n,k-1}^{p,q}(p, pqt) |pt - y_k| \\
 & \quad \times \left(1 + \frac{[n+2]_{p,q}}{[n+1]_{p,q}} p^{-(n+2-k)} \frac{[n+1]_{p,q}}{[k]_{p,q}} pt \right) d_{p,q}t \\
 & \leq \|g'\| \left(1 + p^{-(n+1-k)} \frac{[n+2]_{p,q}}{[k]_{p,q}} \right) \\
 & \quad \times \left\{ [n+1]_{p,q} p^{-(n^2+3n-(k-1)^2-(k-1))/2} \int_0^1 \tilde{b}_{n,k-1}^{p,q}(p, pqt) (pt - y_k)^2 d_{p,q}t \right\}^{1/2} \\
 & = \|g'\| \left(1 + p^{-(n+1-k)} \frac{[n+2]_{p,q}}{[k]_{p,q}} \right) \\
 & \quad \times \left\{ [n+1]_{p,q} p^{-(n^2+3n-(k-1)^2-(k-1))/2} \int_0^1 \tilde{b}_{n,k-1}^{p,q}(p, pqt) (pt - y_k)^2 d_{p,q}t \right\}^{1/2} \\
 & = \|g'\| \left(1 + p^{-(n+1-k)} \frac{[n+2]_{p,q}}{[k]_{p,q}} \right) \left\{ p^{2(n+2-k)} \frac{[k]_{p,q}[k+1]_{p,q}}{[n+2]_{p,q}[n+3]_{p,q}} \right. \\
 & \quad \left. - 2p^{n+2-k} \frac{[k]_{p,q}}{[n+2]_{p,q}} p^{n+2-k} \frac{[k]_{p,q}}{[n+2]_{p,q}} + \left(p^{n+2-k} \frac{[k]_{p,q}}{[n+2]_{p,q}} \right)^2 \right\} \\
 & = \|g'\| \left(1 + p^{-(n+1-k)} \frac{[n+2]_{p,q}}{[k]_{p,q}} \right) p^{n+2-k} \left(\frac{[k]_{p,q}}{[n+2]_{p,q}} \left(\frac{[k+1]_{p,q}}{[n+3]_{p,q}} - \frac{[k]_{p,q}}{[n+2]_{p,q}} \right) \right)^{1/2}
 \end{aligned}$$

Thus, we have

$$|\lambda_{n,k-1}^{p,q}(g) - \lambda_{n+1,k}^{p,q}(g)| = \|g'\| \left(p^{n+2-k} \frac{[k]_{p,q}}{[n+2]_{p,q}} + p \right)$$

$$(2.13) \quad \left(\frac{[n+2]_{p,q}}{[k]_{p,q}} (pq)^k \frac{[n+2-k]_{p,q}}{[n+2]_{p,q}[n+3]_{p,q}} \right)^{1/2}.$$

On the other hand, by (1.1), we have $[n+2]_{p,q} = p^{n+1}[n+2]_{q/p} \geq p^{n+1}[k]_{q/p} = p^{n+2-k}[k]_{p,q}$ for $k = 1, 2, \dots, n+1$. Further $[n+3]_{p,q} = p^{k+1}[n+2-k]_{p,q} + q^{n+2-k}[k+1]_{p,q}$, $k = 1, 2, \dots, n+1$, thus

$$\frac{[n+2-k]_{p,q}}{[n+3]_{p,q}} \leq p^{-(k+1)},$$

moreover $[k]_{p,q} = p^{k-1}[k]_{q/p} \geq p^{k-1}[1]_{q/p} = p^{k-1}$ for $k = 1, 2, \dots, n+1$. Hence, by (2.13), we have

$$(2.14) \quad \begin{aligned} |\lambda_{n,k-1}^{p,q}(g) - \lambda_{n+1,k}^{p,q}(g)| &\leq (1+p)\|g'\| \left((pq)^k p^{-(k-1)} p^{-(k+1)} \right)^{1/2} \\ &= (1+p)\|g'\| \left(\frac{q}{p} \right)^{k/2}. \end{aligned}$$

Because

$$\frac{[n+1-k]_{p,q}}{[n+1]_{p,q}} = p^{-k} \frac{[n+1-k]_{q/p}}{[n+1]_{q/p}} \leq p^{-k}$$

for $k = 0, 1, \dots, n$, and

$$\frac{[k]_{p,q}}{[n+1]_{p,q}} = p^{-(n+1-k)} \frac{[k]_{q/p}}{[n+1]_{q/p}} \leq p^{-(n+1-k)}$$

for $k = 1, 2, \dots, n+1$, we obtain, in view of (2.2), (2.11) and (2.14), that

$$\begin{aligned} &\left| \lambda_{n,k}^{p,q}(g) \frac{[n+1-k]_{p,q}}{[n+1]_{p,q}} p^k + \lambda_{n,k-1}^{p,q}(g) \frac{[k]_{p,q}}{[n+1]_{p,q}} q^{n+1-k} - \lambda_{n+1,k}^{p,q}(g) \right| \\ &\leq \frac{[n+1-k]_{p,q}}{[n+1]_{p,q}} p^k |\lambda_{n,k}^{p,q}(g) - \lambda_{n+1,k}^{p,q}(g)| + \frac{[k]_{p,q}}{[n+1]_{p,q}} q^{n+1-k} |\lambda_{n,k-1}^{p,q}(g) - \lambda_{n+1,k}^{p,q}(g)| \\ &\leq \frac{1+p-q}{1-q} \|g'\| \left(\frac{q}{p} \right)^{n/2} \left(\frac{q}{p} \right)^{3/2} + (1+p)\|g'\| \left(\frac{q}{p} \right)^{n+1-k} \left(\frac{q}{p} \right)^{k/2} \\ &\leq \frac{1+p-q}{1-q} \left(\frac{q}{p} \right)^{n/2} \left(\frac{q}{p} \right)^{3/2} + (1+p)\|g'\| \left(\frac{q}{p} \right)^{(2n-k+2)/2} \\ &\leq \|g'\| \left(\frac{q}{p} \right)^{n/2} \left\{ \frac{1+p-q}{1-q} \left(\frac{q}{p} \right)^{3/2} + \sqrt{\frac{q}{p}} (1+p) \right\}. \end{aligned}$$

This means that we may choose $\beta_n = (q/p)^{n/2}$, $n \geq 1$ (see Theorem 1.1). Then for all $n, m \geq 1$, we have

$$\beta_n + \beta_{n+1} + \dots + \beta_{n+m-1} = \left(\frac{q}{p} \right)^{n/2} + \left(\frac{q}{p} \right)^{(n+1)/2} + \dots + \left(\frac{q}{p} \right)^{(n+m-1)/2}$$

$$= \left(\frac{q}{p}\right)^{n/2} \frac{1 - \left(\frac{q}{p}\right)^{m/2}}{1 - \left(\frac{q}{p}\right)^{1/2}} < \frac{\sqrt{p}}{\sqrt{p} - \sqrt{q}} \left(\frac{q}{p}\right)^{n/2}.$$

Thus we may choose $\alpha_n = (q/p)^{n/2}$, $n \geq 1$. Applying Theorem 1.1, we get the statement of our theorem.

In the next theorem we shall estimate the error $|\tilde{D}_\infty^{p,q}(f; x) - f(x)|$ with the aid of the modulus of continuity (1.4). \square

Theorem 2.2. *For the limit (p, q) -Bernstein-Durrmeyer operator $\tilde{D}_\infty^{p,q}$, we have*

$$|\tilde{D}_\infty^{p,q}(f; x) - f(x)| \leq 2\omega\left(f, \sqrt{\delta_{p,q}(x)}\right)$$

for all $f \in C[0, 1]$ and $x \in [0, 1]$, where

$$\delta_{p,q}(x) = \frac{1}{p^4}(p - q)\{2p^2 + (3p + 1)x + (p^3 - 1)x^2\}.$$

Proof. In view of Lemma 2.3, we have

$$(2.15) \quad \tilde{D}_\infty^{p,q}(1, x) = 1.$$

Further, by Lemma 2.3 and (1.1),

$$\tilde{D}_n^{p,q}(t; x) = \frac{p^{n+1} + p^n q[n]_{q/p} x}{p^{n+1}[n+2]_{q/p}} = \frac{p + q[n]_{q/p} x}{p[n+2]_{q/p}} \rightarrow \frac{p + \frac{q}{1-\frac{q}{p}}x}{p \frac{1}{1-\frac{q}{p}}} = \frac{p - q + qx}{p} =: \tilde{D}_\infty^{p,q}(t; x)$$

as $n \rightarrow \infty$, and analogously, we have

$$\begin{aligned} \tilde{D}_n^{p,q}(t^2; x) &= \frac{p^{2n+2}[2]_{p,q}}{[n+2]_{p,q}[n+3]_{p,q}} + \frac{(2q^2 + qp)p^{n+2}[n]_{p,q}x}{[n+2]_{p,q}[n+3]_{p,q}} \\ &\quad + \frac{q^3[n]_{p,q}[p^2[n]_{p,q}x^2 + p^{n+1}x(1-x)]}{[n+2]_{p,q}[n+3]_{p,q}} \\ &\rightarrow \frac{(p+q)(p-q)^2}{p^3} + \frac{q(p+2q)(p-q)x}{p^3} + \frac{q^3}{p^3} \left\{ x^2 + \frac{p-q}{p}x(1-x) \right\} \\ &=: \tilde{D}_\infty^{p,q}(t^2; x) \end{aligned}$$

as $n \rightarrow \infty$. Then

$$\begin{aligned} &\tilde{D}_\infty^{p,q}((t-x)^2; x) \\ &= \frac{1}{p^3}(p+q)(p-q)^2 + \frac{q^3}{p^4}(p-q)x(1-x) + \frac{1}{p^3}(pq + 2q^2 - 2p^2)(p-q)x \\ &\quad + \frac{1}{p^3}(p^2 - pq - q^2)(p-q)x^2 \\ &\leq \frac{2}{p^2}(p-q) + \frac{1}{p^4}(p-q)x(1-x) + \frac{3}{p^3}(p-q)x + \frac{1}{p}(p-q)x^2 \\ (2.16) &= \delta_{p,q}(x). \end{aligned}$$

For the modulus of continuity (1.4) we have $\omega(f, \lambda\delta) \leq (1 + \lambda)\omega(f, \delta)$, $\lambda \geq 0$. Then

$$|f(t) - f(x)| \leq \omega(f, |t - x|) \leq (1 + \delta^{-1}|t - x|)\omega(f, \delta)$$

for $t, x \in [0, 1]$. Hence, by (2.15), Hölder's inequality and (2.16), we obtain

$$\begin{aligned} & |\tilde{D}_{\infty}^{p,q}(f; x) - f(x)| \\ & \leq \tilde{D}_{\infty}^{p,q}(|f(t) - f(x)|; x) \leq \omega(f, \delta) \left(1 + \delta^{-1} \tilde{D}_{\infty}^{p,q}(|t - x|; x)\right) \\ & \leq \omega(f, \delta) \left\{1 + \delta^{-1} (\tilde{D}_{\infty}^{p,q}((t - x)^2; x))^{1/2}\right\} \leq \omega(f, \delta) \left\{1 + \delta^{-1} \sqrt{\delta_{p,q}(x)}\right\}. \end{aligned}$$

Choosing $\delta = \sqrt{\delta_{p,q}(x)}$, we get the assertion of the theorem. \square

Remark 2.2. If $p = p(q)$ and $q \rightarrow 1$, then Theorem 2.2 implies that $\tilde{D}_{\infty}^{p,q}(f; x)$ converges uniformly to $f(x)$ for $x \in [0, 1]$.

Acknowledgements. The authors are thankful to the reviewers for valuable suggestions leading to overall improvements in the paper.

REFERENCES

1. T. ACAR, A. ARAL and S. A. MOHIUDDINE: *Approximation by bivariate (p, q) -Bernstein-Kantorovich operators*, Iran. J. Sci. Technol. Trans. A Sci., DOI: 10.1007/s40995-016-0045-4.
2. A. ARAL and V. GUPTA: *(p, q) -Type beta functions of second kind*, Adv. Oper. Theory, **1**(1) (2016) 134–146.
3. A. ARAL, V. GUPTA and R. P. AGARWAL: *Applications of q -Calculus in Operator Theory*, Springer, New York, 2013.
4. Q.B. CAI and G. ZHOU: *On (p, q) -analogue of Kantorovich type Bernstein–Stancu–Schurer operators*, Appl. Math. Comput., **276** (2016) 12–20.
5. M. M. DERRIENNIC: *Sur l'approximation de fonctions intégrables sur $[0, 1]$ par des polynômes de Bernstein modifies*, J. Approx. Theory **31** (1981), 325–343.
6. Z. FINTA: *Korovkin type theorem for sequences of operators depending on a parameter*, Demonstratio Math. **48** (3)(2015), 391–403.
7. Z. FINTA: *Approximation by limit q -Bernstein operator*, Acta Univ. Sapientiae, Mathematica **5** (1)(2013), 39–46.
8. V. GUPTA: *(p, q) Genuine Bernstein Durrmeyer operators*, Bull Un Matt. Ital. **9** (3) (2016), 399–409.
9. V. GUPTA: *(p, q) -Szász–Mirakyan–Baskakov operators*, Complex Anal. Operator Theory, (in press), DOI 10.1007/s11785-015-0521-4.
10. V. GUPTA and A. ARAL: *Bernstein Durrmeyer operators based on two parameters*, Facta Universitatis (Niš), Ser. Math. Inform **31** (1) (2016), 79–95.

11. V. GUPTA and R. P. AGARWAL: *Convergence Estimates in Approximation Theory*, Springer, New York, 2014.
12. A. IL'INSKII and S. OSTROVSKA: *Convergence of generalized Bernstein polynomials*, J. Approx. Theory **116** (2002), 100-112.
13. A. LUPAŞ: *A q -analogue of the Bernstein operator*, Babeş-Bolyai University, Seminar on Numerical and Statistical Calculus **9** (1987), 85-92.
14. G. V. MILOVANOVIC, V. GUPTA and N. MALIK: *(p, q) -Beta functions and applications in approximation*, Boletn de la Sociedad Matemtica Mexicana, in press DOI 10.1007/s40590-016-0139-1
15. M. MURSALEEN, K. J. ANSARI and A. KHAN: *On (p, q) -analogue of Bernstein operators*, Appl. Math. Comput. **266** (2015), 874-882.[Erratum: Appl. Math. Comput. **278** (2016) 70–71].
16. M. MURSALEEN, MD. NASIUZZAMAN and A. NURGALI: *Some approximation results on Bernstein-Schurer operators defined by (p, q) -integers*, J. Ineq. Appl. **2015** (2015): 249.
17. H. ORUÇ and G. M. PHILLIPS: *A generalization of the Bernstein polynomials*, Proc. Edinb. Math. Soc. **42** (1999), 403-413.
18. G. M. PHILLIPS: *Bernstein polynomials based on the q -integers*, Ann. Numer. Math. **4** (1997), 511-518.
19. P. N. SADJANG: *On the (p, q) -Gamma and the (p, q) -Beta functions*, arXiv:1506.0739v1. 22 Jun 2015.
20. P. N. SADJANG: *On the fundamental theorem of (p, q) -calculus and some (p, q) -Taylor formulas*, arXiv:1309.3934 [math.QA].
21. V. SAHAI and S. YADAV: *Representations of two parameter quantum algebras and p, q -special functions*, J. Math. Anal. Appl. **335** (2007), 268-279.
22. H. SHARMA: *On Durrmeyer-type generalization of (p, q) -Bernstein operators*, Arab. J. Math. **5** (4)(2016), 239-248.
23. H. WANG and F. MENG: *The rate of convergence of q -Bernstein polynomials for $0 < q < 1$* , J. Approx. Theory **136** (2005), 151-158.

Zoltan Finta
 Babeş-Bolyai University
 Department of Mathematics
 1, M. Kogălniceanu st.
 400084 Cluj-Napoca, Romania
 fzoltan@math.ubbcluj.ro

Vijay Gupta
 Netaji Subhas Institute of Technology
 Department of Mathematics
 Sector 3 Dwarka
 New Delhi 110078, India
 vijaygupta2001@hotmail.com